Analytical Level Crossing Rate and Average Outage Duration for MRC Combiner in Log Normal Shadowed Channels

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Abstract—We provide an analytical approach for the evaluation of important second order statistical parameters, as level crossing rate (LCR) and average outage duration (AOD) of MRC combiner in Log-Normal shadowed channels. New expressions have been derived for LCR and AOD in closed form. The results are valid for an arbitrary number of independent identically distributed diversity branches in the presence of Log-Normal shadowing. The output of MRC combiner has been estimated using Fenton-Wilkinson method. LCR and AOD have been plotted and also tabulated for different number of diversity branches. The results that we provide in this paper are an important tool for measuring the performance of communication links in a log-normal shadowing.

Index Terms – Level Crossing Rate (LCR), Average Outage Duration (AOD), Maximal Ratio Combining (MRC), Log-Normal shadowed (LN), Signal to Noise Ratio (SNR), Independent Identically Distributed (IID)

I. INTRODUCTION

The log-normal distribution is often found to be the most suitable distribution to fit empirical fading channel measurements particularly for the indoor radio propagation environments [9]. The use of log-normal distribution [1], [10] to model shadowing which is random variable (SNR) can’t be expressed in closed form solution for integrations involving in sum of random variables (SNRs) of MRC combiner output. This distribution (PDF) can be approximated by another log normal random variable using Fenton-Wilkinson method [3]. The level crossing rate (LCR) and average outage duration (AOD) are important metrics for evaluating the dynamic performance of diversity systems [11]. LCR and AOD have been considered in [11, 12-16] of different fading channels. These papers have considered Nakagami, Rayleigh, Rician, Rician-Log normal and Rayleigh-Log normal shadowed channels. A method to derive LCR and AOD for the log normal shadowed channel has been shown in [5]. Rice has given a method to find LCR and AOD of a fading channel using the joint PDF of the fading envelope and its time derivative [17]. But traditional Rician approach to estimate LCR and AOD is more complex and not suitable to characterize discrete channels [4, 6, 7].

In this work we have proposed a more reliable method for the statistical estimation of LCR and AOD. We have proposed a new and simple analytical approach to evaluate LCR and AOD of MRC combiner output in Log Normal (LN) shadowed channels. A simple joint CDF based approach for LCR and AOD has been shown. The MRC combiner output has been estimated using Fenton-Wilkinson method. Different values of Means and Variances have been tabulated for different number of diversity branches L. LCR and AOD has been plotted and tabulated for different number of diversity branches L.

II. SYSTEMS AND CHANNEL MODELS

Log-normal Distribution  
A RV $\gamma$ is log-normal, i.e. $\gamma \sim LN (\mu, \sigma^2)$, if and only if $\ln(\gamma) \sim N(\mu, \sigma^2)$. A log-normal RV has the PDF

$$p(\gamma) = \frac{1}{\gamma \sigma \sqrt{2\pi}} e^{-\frac{(\ln \gamma - \mu)^2}{2 \sigma^2}}$$  \hspace{1cm} (1)

For any $\sigma^2 > 0$. The expected value of $\gamma$ is $$E(\gamma) = e^{(\mu + 0.5 \sigma^2)}$$

And the variance of $\gamma$ is $$Var(\gamma) = e^{(\mu^2 + \sigma^2)} - e^{(2 \mu + \sigma^2)}$$

Where $\mu = \ln(10) = 4.3429$. $\mu$ (dB) is the mean of $10 \log_{10} \gamma$. $\sigma$ (dB) is standard deviation of $10 \log_{10} \gamma$.

CDF of log-normal RV

$$P(\gamma) = Q \left( \frac{\mu - 10 \log_{10} \gamma}{\sigma} \right)$$  \hspace{1cm} (2)

Where $Q(\cdot)$ is one dimensional standard Gaussian Q function.

III. MAXIMAL RATIO COMBINING

The total SNR at the output of the MRC combiner is given by:

$$\gamma_{MRC} = \sum_{i=1}^{L} \gamma_i$$  \hspace{1cm} (3)
Where \( L \) is number of IID branches.

Using Fenton-Wilkinson method, PDF of MRC combiner output is

\[
p(Y_{\text{MRC}}) = \frac{\xi}{\sigma_M Y_{\text{MRC}}} \sqrt{2\pi} e^{-\left(\frac{(10\log_{10} Y_{\text{MRC}} - B_M)^2}{2\sigma_M^2}\right)}
\]

And CDF of MRC combiner output is

\[
P(y) = Q\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right)
\]

### Table I \( \mu_M \) and \( \sigma_M \) for different number of diversity branches [8]

<table>
<thead>
<tr>
<th>Number of diversity branches ( L )</th>
<th>( \mu_M )</th>
<th>( \sigma_M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.04</td>
<td>0.85</td>
</tr>
<tr>
<td>4</td>
<td>2.70</td>
<td>0.65</td>
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<tr>
<td>6</td>
<td>3.51</td>
<td>0.55</td>
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<td>8</td>
<td>4.05</td>
<td>0.48</td>
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<tr>
<td>10</td>
<td>4.51</td>
<td>0.44</td>
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<td>20</td>
<td>6.35</td>
<td>0.31</td>
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<td>7.76</td>
<td>0.26</td>
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<tr>
<td>50</td>
<td>10.01</td>
<td>0.20</td>
</tr>
</tbody>
</table>

### IV. LEVEL CROSSING RATE AND AVERAGE OUTAGE DURATION

The AOD, \( T_{\gamma th} \) (in seconds) is a measure of how long, on the average, the system remains in the outage state [1].

\[
T_{\gamma th} = \frac{P_{\text{out}}}{N_{\gamma th}}
\]

Where \( P_{\text{out}} \) is outage probability.

The envelope level crossing rate \( N_{\gamma th} \) is defined as the expected rate (in crossings per second) at which the signal envelope crosses the level \( \gamma th \) in the downward direction [2]

\[
N_{\gamma th} = \int_0^\infty \tilde{y} f_{\tilde{\gamma} \tilde{y}}(\gamma th, \tilde{y}) d\tilde{y}
\]

Where \( f_{\tilde{\gamma} \tilde{y}}(\gamma th, \tilde{y}) \) is the joint PDF of \( \tilde{y} \) and its time derivative. Above definitions are Rician approach to estimate LCR and AOD. But this approach is not suitable for discrete channels. So for discrete sampled random process, LCR can be defined as the rate at which the envelope \( y \) crosses a certain threshold \( \gamma th \) in the positive or in the negative direction

\[
N_y = \frac{P(\gamma 1 \leq y, \gamma 2 \geq \gamma th)}{T}
\]

Where \( \gamma 1 \pm \gamma(t) \)and \( \gamma 1 \pm \gamma(t + T) \) is second order statistics and \( T \) denotes sampling period.

Here \( \gamma 1 \) and \( \gamma 2 \) are uncorrelated IID random variables so their CDF is same as shown in (6).

Noting here

\[
P(\gamma 1 \leq y, \gamma 2 \geq \gamma th) = P(\gamma 1 \leq \gamma th) - P(\gamma 1 \leq \gamma th, \gamma th \geq \gamma 2)
\]

Or

\[
P(\gamma 1 \leq u, \gamma 2 \geq u) = CDF \ of \ \gamma 1 - Joint \ CDF \ of \ \gamma 1 \ and \ \gamma 2
\]

CDF of \( \gamma 1 \) for \( L \) diversity branches

\[
P(\gamma 1 \leq \gamma th) = Q\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right)
\]

And Joint CDF of \( \gamma 1 \) and \( \gamma 2 \) for \( L \) diversity branches

\[
P(\gamma 1 \leq \gamma th, \gamma 2 \geq \gamma th) = Q\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right) Q\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right)
\]

Using (8), (9) and (10) in (7), LCR of MRC combiner output in log normal shadowed channel is

\[
N_{\gamma th} = \frac{Q\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right) - Q^2\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right)}{T}
\]

AOD is defined as

\[
T_{\gamma th} = \frac{P(\gamma \leq \gamma th)}{N_{\gamma th}}
\]

Using (9) and (11) in (12) we have AOD of MRC combiner output in log normal shadowed channel is

\[
T_{\gamma th} = \frac{T \cdot Q\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right)}{Q\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right) - Q^2\left(\frac{\mu_M - 10\log_{10} Y_{\text{MRC}}}{\sigma_M}\right)}
\]
REFERENCES

V. CONCLUSION
This paper has established a process for estimating the LCR and AOD for MRC combiner output in log normal shadowed channel. Here we have used Table I which was given in [8]. LCR and AOD has been plotted and tabulated as well for different number of diversity branches L. For low threshold level, LCR decreases with increase in number of diversity branches L while for high threshold level (SNR), LCR increases with increase in number of diversity branches L. For low threshold level (SNR), AOD increases with increase in number of diversity branches L while for high threshold level, AOD decreases with increase in number of diversity branches L. So we can conclude that MRC combining improves system’s performance in terms of LCR and AOD in log normal shadowed channel.
Appendix (Table II & Table III)

**Table II Level Crossing Rate of MRC combiner output**

<table>
<thead>
<tr>
<th>$\gamma_{th}$</th>
<th>$T_{\gamma_{th}}$</th>
<th>Number of diversitiy branches $L$</th>
<th>$T_{\gamma_{th}}$</th>
<th>Number of diversitiy branches $L$</th>
<th>$T_{\gamma_{th}}$</th>
<th>Number of diversitiy branches $L$</th>
<th>$T_{\gamma_{th}}$</th>
<th>Number of diversitiy branches $L$</th>
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**Table III Average Outage Duration of MRC combiner output**

<table>
<thead>
<tr>
<th>$\gamma_{th}$</th>
<th>$T_{\gamma_{th}}$</th>
<th>Number of diversitiy branches $L$</th>
<th>$T_{\gamma_{th}}$</th>
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<td>$= -5 dB$</td>
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